Robust Multigrid with Cell-based Coarsening by a Factor of Three

Joel E. Dendy, David J. Moulton, T-5

Multigrid methods are well established as one of the most effi-cient algorithms for the solution of the discrete linear systems that arise from models of diffusive phenomena (e.g., heat conduction, neutron diffusion, single-phase saturated flow). In particular, their solution cost grows only linearly with the size of the discrete system. These methods achieve this optimal scaling through the recursive use of successively coarser discrete problems (i.e., a sequence of coarse-grid discrete operators) in conjunction with smoothing on each level (e.g., a single Gauss-Seidel iteration on each level) to damp the highly oscillatory errors associated with each grid.

However, developing a robust multigrid algorithm with optimal algorithmic scaling for models with highly discontinuous and anisotropic coefficients is a significant challenge. Here a naive approach based on creating a hierarchy of discrete problems through rediscretization at coarser resolutions leads to a fragile algorithm with a convergence rate that depends on the magnitude of the jumps in the diffusion coefficient. Thus, a fundamental advance in multigrid methods was the development of *Black Box* methods that use the fine-scale discrete model to construct, through a variational principle, the successively coarser coarse-grid operators [1]. In fact, robust methods for both structured grids, such as Black Box Multigrid (BoxMG), and unstructured grids, such as Algebraic Multigrid (AMG), use this variational approach.

Despite this powerful technique, additional constraints on key elements of the multigrid algorithms may complicate its design. For example, it may be desirable to preserve the cell structure in the hierarchy of discrete operators. This desire often arises with cell-based finite volume methods, and with cell-based adaptive mesh refinement. Unfortunately, a straightforward application of the BoxMG method [1], when coarsening by a factor of two, does not preserve the cell-based structure.

To preserve the cell-based structure and maintain a fixed operator complexity we proposed a variant of BoxMG that uses coarsening by a factor of three [2]. This cell-based coarsening by three is shown schematically in Fig. 1(top, left), where the coarse-grid cells are shown by the darker lines, and coarse-grid cell-center unknowns are shown as the darker circles. This coarsening strategy ensures that the cell-centered unknowns are nested on coarser grids and exposes interesting connections between BoxMG, smoothed-aggregation based AMG, and Multiscale Finite Elements. The key components of the new BoxMG algorithm follows.

- **Galerkin Coarse-Grid Operator.** Variational coarsening, employing the Galerkin coarse-grid operator, can be shown to be optimal in the sense that it minimizes the error in the range of interpolation. Moreover, this approach makes no assumption about the form of the coarse-scale model. This feature makes it ideal for problems with fine-scale spatial structure in the coefficients, as rediscretization fails to capture the influence of this structure at coarser scales.

- **Operator-Induced Interpolation.** We developed a generalization of the operator-induced interpolation methodology for coarsening by two, to coarsening by three. First we split the points into coarse and fine points, as shown in Fig.1(top,right) with interpolation of coarse points given by the identity. The fine points are then split into two types: fine points that are embedded in coarse-grid lines (shown as small boxes in Fig.1, bottom, left), and fine points that are in the interior of a coarse-grid cell (shown as small hexagons in Fig.1, bottom, right). We then average the discrete operator at these fine-grid points to define the entries in the corresponding block matrices (4 x 4 and 2 x 2 systems) that are inverted to yield the interpolation weights.

• **Pattern Relaxation.** Although standard smoothing techniques, such as colored Gauss-Seidel are applicable here, in many situations these methods require an extra sweep of points at the top and right boundaries. To address this weakness we developed a new relaxation technique dubbed *pattern relaxation*. This smoother is motivated by the structure of the interpolation operator and uses a four-color block Gauss-Seidel scheme that corresponds to the blocks shown schematically in Fig.1. An interesting aspect of pattern relaxation is its potential for coarsening by larger factors *n*, as is common in many multiscale methods.

To demonstrate the robustness of the proposed BoxMG method with respect to discontinuous diffusion and removal coefficients, different types of boundary conditions, and grid dimensions that are not optimal multiples of three, we considered a suite of test problems. For example, we considered a simple Poisson problem with favorable grid dimensions (3*m* and 3*m*+2), as well as unfavorable grid dimensions (3*m*+1), where *m* is an integer grid-size parameter. Within each of these categories the convergence rate was shown to be independent of *m*. To explore discontinuous diffusion coefficients we considered a thin layer problem as well as a variant of the classic checkerboard problem. Performance in these cases was similar to the simple Poisson problem with an average convergence factor for the largest problems of approximately 0.12. In addition, we demonstrated the effectiveness of the algorithm for mixed boundary conditions.

We have developed a robust multigrid method that uses coarsening by three to preserve the cell-based structure of the fine-grid problem. In this method we generalized the concept of operator-induced interpolation and developed a new relaxation method dubbed pattern relaxation. In the future we will investigate the extension of pattern relaxation to anisotropic problems. In addition, we will explore the connections of this new multigrid method with smoothed-aggregation based AMG and multiscale methods.
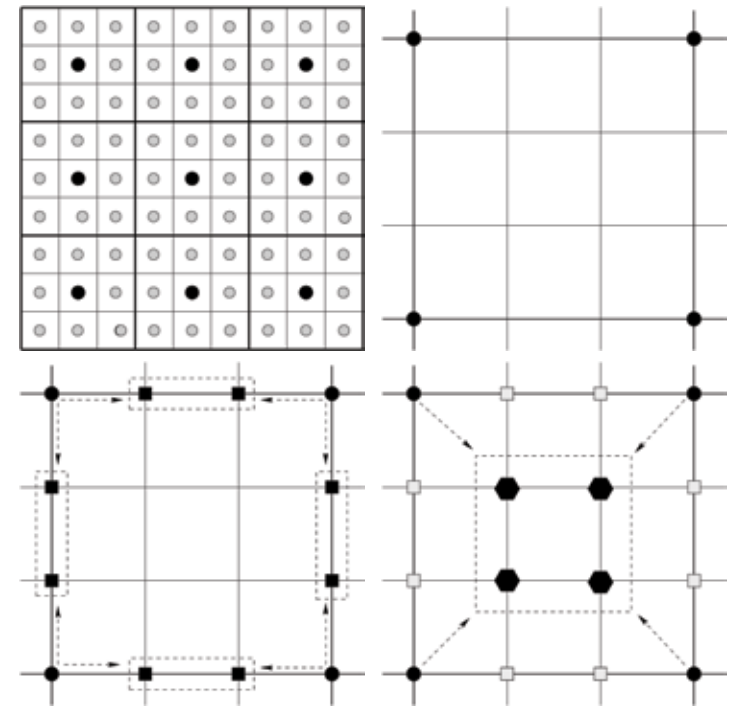

Fig 1. Cell-based coarsening by three is shown in the upper left. The structure of the operator-induced interpolation is shown schematically by the sequence of figures on a coarse-grid cell: injection of nested points in the upper right, followed by a block update of points embedded in coarse-grid lines (lower left) and finally the block update of fine-grid points contained in a coarse-grid cell (lower right). This operator-induced interpolation is critical to the method's robustness.

For more information contact David Moulton at moulton@lanl.gov.

[1] J.E. Dendy, *J. Comput. Phys.,* **48**, 366, (1982).
[2] J.E. Dendy, J.D. Moutlton, *Numer. Linear Algebr.* (to appear) (2010).